Classical Boundary-value Problem in Riemannian Quantum Gravity and Taub-Bolt-anti-de Sitter Geometries

M M Akbar*

Department of Applied Mathematics and Theoretical Physics,
Centre for Mathematical Sciences,
University of Cambridge,
Wilberforce Road,
Cambridge CB3 0WA,
U.K.

February 7, 2008

Abstract

For an $SU(2) \times U(1)$-invariant S^3 boundary the classical Dirichlet problem of Riemannian quantum gravity is studied for positive-definite regular solutions of the Einstein equations with a negative cosmological constant within biaxial Bianchi-IX metrics containing bolts, i.e., within the family of Taub-Bolt-anti-de Sitter (Taub-Bolt-AdS) metrics. Such metrics are obtained from the two-parameter Taub-NUT-anti-de Sitter family. The condition of regularity requires them to have only one free parameter (L) and constrains L to take values within a narrow range; the other parameter is determined as a double-valued function of L and hence there is a bifurcation within the family. We found that *any* axially symmetric S^3-boundary can be filled in with at least one solution coming from each of these two branches despite the severe limit on the permissible values of L. The number of infilling solutions can be one, three or five and they appear or disappear catastrophically in pairs as the values of the two radii of S^3 are varied. The solutions occur simultaneously in both branches and hence the total number of independent infillings is two, six or ten. We further showed that when the two radii are of the same order and large the number of solutions is two. In the isotropic limit this holds for small radii as well. These results are to be contrasted with the one-parameter self-dual Taub-NUT-AdS infilling solutions of the same boundary-value problem studied previously.

1 Introduction

When evaluating the path integral of Riemannian quantum gravity one often requires an answer to the following Dirichlet problem: for a given closed manifold Σ with metric h_{ij} what are the

---

*Email:M.M.Akbar@damtp.cam.ac.uk

regular compact solutions $(\mathcal{M}, g_{\mu\nu})$ such that $\partial\mathcal{M} = \Sigma$ and $g_{\mu\nu}|_{\partial\mathcal{M}} = h_{ij}$ and $g_{\mu\nu}$ satisfies the Einstein field equations with appropriate matter fields or a cosmological constant? Such classical solutions $(\mathcal{M}, g_{\mu\nu})$ are often referred to as infilling geometries of the 'boundary' (Σ, h_{ij}). They give semi-classical approximations to the path integral and provide valuable insights into the nature of quantum gravity.

The general problem of finding all infilling solutions without any restrictions on the possible forms of infilling metrics is a difficult problem and often non-solvable. The boundary conditions and the condition for regularity make the problem more non-trivial than just finding Euclidean metrics solving Einstein's equations. Fortunately, one often restricts oneself to rather symmetrical systems because of physical or technical considerations. For example, in cosmological problems one usually assumes (Σ, h_{ij}) to be a homogeneous manifold invariant under the transitive action of some Lie group H, and assumes that a possible infilling 4-metric is of cohomogeneity one under the group action of H. However, the generic orbits of the infilling 4-metrics can be assumed to have less symmetry and invariant under the group action of $H' \subseteq H$, provided H' expands to H on the boundary (Σ, h_{ij}). In any case the assumption of cohomogeneity (under the group action of H or H') reduces the Dirichlet problem to a set of ordinary differential equations (assuming, of course, no matter is present and only a cosmological constant term exists) to be solved subject to the boundary data and the condition of regularity in the interior. However, if complete solutions of such cohomogeneity one metrics are known in advance, the problem can be reduced to that of isometric embedding of a given manifold (Σ, h_{ij}) in a manifold $(\tilde{\mathcal{M}}, g_{\mu\nu})$ one dimension higher. The infilling geometry $(\mathcal{M}, g_{\mu\nu})$ is then the nonsingular compact part(s) of the manifold $\tilde{\mathcal{M}}$ cut by the codimension-one hypersurface Σ. Consider the archetypal example of an S^3 boundary with the canonical round metric on it. For any value of the (negative) cosmological constant, it can be filled in by parts of H^4 (with the standard metric on it). On the other hand it is only possible to embed the S^3 in a round S^4 if its radius is not greater than the 4-radius of the S^4 (assuming a fixed positive cosmological constant which determines the radius of S^4). Note that in either of these infillings the generic orbits strictly have the symmetry of the boundary. However, as mentioned above, one can in principle consider infillings which are less symmetric in the interior. For example, as shown in [1] as part of a more general boundary-value problem, any round S^3 can be filled in with a regular biaxial Bianchi-IX solution with four-ball topology in the presence of a negative cosmological constant.

In this paper we consider the boundary-value problem for an S^3-boundary endowed with the following 3-metric

$$ds^2 = a^2(\sigma_1{}^2 + \sigma_2{}^2) + b^2\sigma_3{}^2, \tag{1.1}$$

where σ_i are the left invariant one forms of $SU(2)$ and hence the 3-boundary is invariant under the group-action of $SU(2) \times U(1) \sim U(2)$. Such squashed S^3's are known as Berger spheres [19]. Following the discussion in the previous paragraph, we naturally want to find solutions within the class of cohomogeneity one metrics admitting $SU(2) \times U(1)$ action, i.e., within the family of positive definite biaxial Bianchi-IX Einstein metrics. The general solution of such metrics is given by the two-parameter Taub-NUT-(anti-)de Sitter family. When one imposes regularity at the origin, one obtains two one-parameter families of metrics: the self-dual Taub-NUT-(anti-)de Sitter and the Taub-Bolt-(anti-)de Sitter. The former metric has a nut at the centre and has a self-dual Weyl curvature tensor. The latter contains a bolt – a singular orbit – corresponding to

2

the two-dimensional fixed-point set of the $U(1)$-action which is an S^2.

The Dirichlet problem within the self-dual Taub-NUT-AdS space was studied in [1]. This lead to a constraint on the possible values of the two radii (a, b) of the Berger sphere in the form of a non-linear inequality between them.[1] In this paper the problem will be studied for the Taub-Bolt-AdS family. As we will see in Section 2.2 the condition of regularity of the bolt constrains the free parameter (L) to take values within a very narrow range on the real line with a two-fold degeneracy in the other parameter (M) which is otherwise determined by L. This is a rather stringent constraint compared to the self-dual Taub-NUT-(anti-)de Sitter for which there is no such *a priori* restriction on the values of the free parameter for regularity. In this case, therefore, one would naturally expect some stronger restrictions on the two radii of the Berger sphere for it to qualify for a Taub-Bolt-AdS infilling. However, this is not the case as we will see below.

Following the AdS/CFT correspondence [17] the Taub-Bolt-AdS and self-dual Taub-NUT-AdS spaces have recently garnered much attention [6, 7, 9, 10, 12, 18]. In particular it was shown in [6, 12] that a thermal phase transition occurs taking self-dual Taub-NUT-AdS to Taub-Bolt-AdS in much the same way the Hawking-Page phase transition occurs from hot AdS space to Schwarzschild-AdS [13]. In studying the semi-classical transition amplitude of the process the action is first calculated for a common finite boundary which encloses the compact parts of the self-dual Taub-NUT-AdS and the Taub-Bolt-AdS space such that they induce the same metric on this common boundary. This common boundary is then taken to infinity to give the transition amplitude from the complete self-dual Taub-NUT-AdS to the complete Taub-Bolt-AdS spaces. Since the boundary is ultimately taken to infinity, the two infilling spaces for the finite boundary need not induce exactly the same metric on the boundary – an approximate matching is sufficient if that approximation is later justified when the boundary is taken to infinity. In this paper we demonstrate that a finite boundary formulation of the Dirichlet problem for the Taub-Bolt-AdS space is in fact highly non-trivial. This has not been addressed rigorously before.[2] Among other results, we find that a typical axially symmetric S^3 boundary admits multiple Taub-Bolt-AdS solutions. One can compare the situation with Schwarzschild-AdS in a finite isothermal cavity in which case the boundary is an $S^1 \times S^2$. Apart from the periodically identified AdS space there are either two or zero AdS-Schwarzschild infilling solutions depending on the boundary data [4]. For a biaxial S^3 boundary, which is a non-trivial S^1 bundle over S^2, we find that the number of independent Taub-Bolt-AdS solutions can be as high as ten. Finding explicit solutions and their possible implications are left for future work.

2   Positive-definite biaxial Binachi-IX Einstein metrics

The Euclidean family of Taub-NUT-(anti-)de Sitter metrics is given by [5, 8, 11]:

$$ds^2 = \frac{\rho^2 - L^2}{\Delta} d\rho^2 + \frac{4L^2 \Delta}{\rho^2 - L^2}(d\psi + \cos\theta d\phi)^2 + (\rho^2 - L^2)(d\theta^2 + \sin^2\theta d\phi^2) \qquad (2.1)$$

---

[1]In order to avoid confusion due to divergent conventions in the literature, we will reserve Taub-NUT-(anti-)de Sitter for the the whole two parameter family and would call the one-parameter family containing (regular) "nuts" as self-dual Taub-NUT-(anti-)de Sitter family as the latter has a self-dual Weyl tensor.

[2]Some investigations of the similar boundary-value problem with the Taub-Bolt-de Sitter metric were made in the context of quantum cosmology in [16].

where

$$\Delta = \rho^2 - 2M\rho + L^2 + \Lambda(L^4 + 2L^2\rho^2 - \frac{1}{3}\rho^4). \tag{2.2}$$

Here L and M are the two parameters and $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \psi \leq 4\pi/k$ ($k \in \mathbb{Z}$). When $k = 1$, the surfaces of constant ρ are topologically S^3. These metrics are Einstein, i.e., they satisfy the Einstein equation with a cosmological constant, $R_{\mu\nu} = \Lambda g_{\mu\nu}$. They are axially symmetric, of Bianchi type IX, i.e., of the form $ds^2 = N(r)^2 dr^2 + a^2(r)(\sigma_1{}^2 + \sigma_2{}^2) + b^2(r)\sigma_3{}^2$, where σ_i are left-invariant one forms on S^3 and hence are invariant under the group action of $SU(2) \times U(1)$. In the literature there has been considerable interest in biaxial Bianchi type IX metrics for both Lorentzian and Riemannian signatures.

The general form (2.1) is only valid for a coordinate patch for which $\Delta \neq 0$. Because it is quartic, $\Delta(\rho)$ will have four roots – the so called "bolts" (two-spheres of constant radii) of the metric (2.1) which are the invariant points of the Killing vector $\partial/\partial\psi$. However, if Δ has a root at $\rho = |L|$, the fixed points there are zero dimensional and such fixed points are called "nuts" [15]. In general, bolts and nuts are not regular points of the metric. The metric can be made regular at a bolt or a nut provided it closes smoothly. For a bolt this is possible if the following condition is satisfied [22]

$$\frac{d}{d\rho}\left(\frac{\Delta}{\rho^2 - L^2}\right)_{(\rho = \rho_{bolt})} = \frac{1}{2kL}, \tag{2.3}$$

which amounts to imposing a relation between L and M. This is also a necessary condition for a regular nut though it is not sufficient to guarantee regularity. (Near a nut regularity requires the metric to approach the flat metric on $\mathbb{E}^4$ which is not guaranteed *a priori* by Eq.(2.3).)

2.1   Self-dual Taub-NUT-(anti-)de Sitter

A zero-dimensional fixed point-set of the $U(1)$-action occurs at $\rho = |L|$, if $\Delta = 0$, i.e., if

$$M = \pm L(1 + \frac{4}{3}\Lambda L^2), \tag{2.4}$$

which is precisely the condition for (anti-)self-duality of Weyl tensor of the metric (2.1), as one can check. It is easy to check that for $k = 1$ the metric approaches the flat metric on $\mathbb{E}^4$ guaranteeing that the metric at the nut is smooth satisfying (2.3) trivially. The metric is then well-defined for the range of ρ starting from the nut until it encounters another zero of Δ. For more details see [1]. Assuming L positive and choosing the positive sign for M, Δ simplifies:

$$\Delta = (\rho - L)^2 - \frac{1}{3}\Lambda(\rho + 3L)(\rho - L)^3. \tag{2.5}$$

For the purpose of the present paper, it is important to note that the condition of regularity (2.3) has determined M in terms of L uniquely (modulo sign) reducing the resulting self-dual Taub-NUT-(anti-)de Sitter metrics effectively to a one-parameter family.

2.2   Taub-Bolt-(anti-)de Sitter

For the purpose of exposition consider the $\Lambda > 0$ case first. Without any loss of generality assume ρ and L are positive. Since a bolt occurs at a zero of Δ, one can find M in the following way.

Suppose the bolt occurs at $\rho = \rho_{bolt}$, then

$$M = \frac{1}{6} \frac{3\,\rho_{bolt}{}^2 - \Lambda\,\rho_{bolt}{}^4 + 3\,L^2 + 3\,\Lambda\,L^4 + 6\,\Lambda\,L^2\rho_{bolt}{}^2}{\rho_{bolt}}. \tag{2.6}$$

By definition $\rho_{bolt} > L$. The regularity condition (2.3) then gives (for $k = 1$)

$$\frac{-\Lambda\,\rho_{bolt}{}^2 + L^2\Lambda + 1}{\rho_{bolt}} = \frac{1}{2L}. \tag{2.7}$$

This is quadratic in ρ_{bolt} and can have only one positive root. It is easy to verify that this root will always occur within the open interval (L, ∞). Corresponding to this value of ρ_{bolt}, M is determined uniquely by (2.6). Thus, for $\Lambda > 0$ the bolt can be made regular for *any* value of L provided one fixes M accordingly, i.e., through (2.6). Therefore the imposition of condition (2.3) has reduced the family to a truly one-parameter family.

In the case of negative cosmological constant the arguments are slightly subtle. Denoting $-\Lambda = \lambda (> 0)$

$$M = \frac{1}{6} \frac{3\,\rho_{bolt}{}^2 + \lambda\,\rho_{bolt}{}^4 + 3\,L^2 - 3\,\lambda\,L^4 - 6\,\lambda\,L^2\rho_{bolt}{}^2}{\rho_{bolt}}. \tag{2.8}$$

The regularity condition (2.3) (for $k = 1$) then gives:

$$\frac{\lambda\,\rho_{bolt}{}^2 - L^2\lambda + 1}{\rho_{bolt}} = \frac{1}{2L} \tag{2.9}$$

which can be rearranged into the more illuminating form:

$$2\,L\lambda\,\left(\rho_{bolt}{}^2 - L^2\right) - (\rho_{bolt} - 2L) = 0. \tag{2.10}$$

Since $\rho_{bolt} > L$, this clearly requires $\rho_{bolt} > 2L$. This equation can be solved to locate the bolt [6]. The bolt is located at either

$$\rho_{bolt} = \frac{1}{4} \frac{1 - \sqrt{1 + 16\,\lambda^2 L^4 - 16\,L^2\lambda}}{\lambda\,L} \tag{2.11}$$

or at

$$\rho_{bolt} = \frac{1}{4} \frac{1 + \sqrt{1 + 16\,\lambda^2 L^4 - 16\,L^2\lambda}}{\lambda\,L} \tag{2.12}$$

provided the quantity under the square root is non-negative. This last requirement, which automatically guarantees that $\rho_{bolt} > 2L$ as one can check, restricts L :

$$\lambda\,L^2 \le \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right) \quad (\sim 0.066987298). \tag{2.13}$$

Therefore if L is within this range the bolt is regular, i.e., at the bolt the metric is the product metric of $\mathbb{E}^2$ and an S^2 of constant radius. The complete metric ($\rho_{bolt} \le \rho < \infty$) is defined over a manifold having the topology of $\mathbb{C}P^2$. Readers interested in more details about the global properties of Taub-Bolt-AdS space are referred to [6, 12].

One can now find M:

$$M = \frac{1}{96} \frac{1 \pm \sqrt{1 + 16\,\lambda^2 L^4 - 16\,\lambda\,L^2}\,\left(32\,\lambda^2 L^4 - 8\,\lambda\,L^2 - 1\right)}{\lambda^2 L^3}. \tag{2.14}$$

The positive and negative signs correspond to the first and second values of ρ_{bolt} above given by Eq. (2.11) and Eq.(2.12) respectively. Eq.(2.14) gives M in terms of L and thus reduces the family from two parameter to one-parameter in the sense that the parameter takes values freely albeit within a range. However, note that these two M are not related by a sign and hence they are not just "orientations" as in the case of self-dual Taub-NUT-(anti-)de Sitter family [1]. This one-parameter family of regular Taub-Bolt-AdS metrics therefore admits two bifurcated sub-families. For a given L within the permissible range, there are therefore two regular Taub-Bolt-AdS metrics.

3   Infilling Taub-Bolt-AdS geometries

We first briefly recall the results obtained in [1] for self-dual Taub-NUT-(anti-)de Sitter solutions of the same boundary-value problem. The problem of finding solutions for a given S^3 specified by its two radii (a, b) amounted to solving two algebraic equations in ρ and L:

$$a^2 - \rho^2 + L^2 = 0, \tag{3.1}$$

and

$$b^2(\rho^2 - L^2) - 4 L^2 \left((\rho - L)^2 - \frac{1}{3} \Lambda \, (\rho + 3 L) \, (\rho - L)^3 \right) = 0. \tag{3.2}$$

This system admits the discrete symmetry $(\rho, L) \leftrightarrow (-\rho, -L)$. With simple substitutions that preserve this symmetry, the above system can be reduced to an algebraic equation of degree three and hence in general there are three, modulo orientation, complex-valued infilling solutions for arbitrary (a, b) for which explicit solutions can be obtained. However, to determine which of them are real infilling solutions one needs to be more careful. One finds that there is a unique real, positive-definite self-dual Taub-NUT-anti-de Sitter solution on the four-ball bounded by a given Berger-sphere of radii (a, b), if and only if

$$b^2 < \frac{1}{3}a^2 \, (2a^2|\Lambda| + 3). \tag{3.3}$$

If this inequality is not satisfied by the radii (a, b), there is no real solution at all. It is not, however, a coincidence that the infilling solution is unique in this case. This is because the set of ordinary differential equations arising from the Einstein equations can be converted into first order equations by applying the condition of self-duality of the Weyl tensor. The boundary data (a, b) then fix their first derivatives uniquely at the boundary and hence evolution is unique. The algebraic condition (3.3) then gives us the condition on the boundary data for which the two radii of the evolving nested Berger spheres can go to zero smoothly.

The algebraic system determining whether an S^3 boundary can be filled in classically with a regular Taub-Bolt-AdS geometry similarly consists of the following two equations in ρ and L:

$$a^2 - \rho^2 + L^2 = 0 \tag{3.4}$$

and

$$b^2 \, (\rho^2 - L^2) - 4 L^2 \left(\rho^2 - 2M\rho + L^2 - \lambda(L^4 + 2L^2\rho^2 - \frac{1}{3}\rho^4) \right) = 0 \tag{3.5}$$

6

where M is given by one of the two choices for corresponding to the two signs in (2.14). As before $\lambda(=-\Lambda)$ is positive. However, compared to the self-dual Taub-NUT-(anti-)de Sitter problem, Eqs.(3.4)-(3.5) are slightly unpromising at first sight. Getting rid of the square roots one would be left with a polynomial equation of higher degree (degree seven, Eq.(3.7) below) than we know how to solve analytically. However, a surprising result follows if one looks carefully:

Theorem: *For* **any** *boundary data* (a, b), *where a and b are the radii of two equal and one unequal axes of a squashed* S^3 *respectively, there exist an odd number of regular Taub-Bolt-AdS solutions bounded by the* S^3 *for either choice of M (and hence, in total, an even number of regular bolt solutions).*

Proof: Denote (a^2, b^2) by (A, B). With the help of the cosmological constant rescale all quantities so that they are dimensionless:

$$A\lambda, \; B\lambda \quad \rightarrow \quad A, \; B \text{ etc.} \tag{3.6}$$

Solve Eq.(3.4) for ρ and substitute into Eq.(3.5). Then, by suitable squaring, one may obtain, *for either choice of M given by Eq.(2.14)*, the following seventh-degree equation for $\lambda L^2 \equiv p$, in which no square roots appear:

$$g(p) := a_7\, p^7 + a_6\, p^6 + a_5\, p^5 + a_4\, p^4 + a_3\, p^3 + a_2\, p^2 + a_1\, p + a_0 = 0 \tag{3.7}$$

and where

$$a_7 \;=\; 8192\, A \left(-6\, AB + 4\, A^3 + 6\, A^2 - 1\right) \left(2\, A^2 + 3\, A - 3\, B\right),$$

$$\begin{aligned}
a_6 \;=\; &-128\, (216\, A^3 - 3456\, BA^3 + 2304\, A^5 - 27 + 128\, A^7 - 288\, BA^2 - 1152\, BA^4 + 36\, A^2 \\
&-36\, BA - 1152\, A^3 B^2 + 1840\, A^4 + 576\, BA^5 + 1728\, A^2 B^2 + 960\, A^6),
\end{aligned}$$

$$\begin{aligned}
a_5 \;=\; &4\, (13824\, A^6 + 20736\, A^2 B^2 + 1728\, A^3 - 4320\, BA + 4320\, A^2 + 27648\, BA^4 + 81 + 3072\, A^7 \\
&+1728\, A + 256\, A^8 + 41472\, BA^5 + 1728\, BA^2 + 28160\, A^5 + 9216\, BA^6 + 20448\, A^4 \\
&-55296\, A^3 B^2 - 39168\, BA^3),
\end{aligned}$$

$$\begin{aligned}
a_4 \;=\; &-8\, A \left(3240\, AB + 10368\, A^4 B - 1440\, A^3 - 81 + 5184\, A^3 B^2 + 10752\, A^3 B + 72\, A^2 B\right. \\
&-2176\, A^2 - 240\, A^4 - 144\, AB^2 - 10368\, A^2 B^2 + 3456\, A^4 B^2 + 3456\, A^5 B + 24\, B \\
&\left.-1728\, A^2 B^3 - 456\, A + 384\, A^6 B\right),
\end{aligned}$$

$$\begin{aligned}
a_3 \;=\; &4\, A(-72\, AB + 1728\, A^3 B^3 + 24\, A^2 - 720\, A^3 B + 864\, A^5 B^2 + 288\, A^2 B^2 - 2160\, A^2 B \\
&+5184\, A^4 B^2 + 7776\, A^3 B^2 + 270\, AB^2 + 36\, B - 2592\, A^2 B^3 + 45\, A + 28\, A^3),
\end{aligned}$$

$$a_2 \;=\; -8\, A^2 \left(-27\, B + 648\, A^2 B^3 + 9\, AB - 135\, AB^2 + 216\, A^3 B^3 + 2\, A^3 + 18\, A + 12\, A^2\right),$$

$$a_1 \;=\; 3\, A^2 B \left(108\, A^2 B^3 + 24\, A + 8\, A^2 - 3\, B\right),$$

$$a_0 \;=\; -9\, A^3 B^2. \tag{3.8}$$

Recall that regular bolt is possible if $p \in \left(0, \frac{2-\sqrt{3}}{4}\right]$. We want to prove the existence of an odd number of roots of Eq.(3.7) within this interval for arbitrary boundary data (A, B) (and, automatically, for either choice of M).

Since $g(p)$ is of odd degree there will be at least one real root. Recall that for any function $f(x)$, if $f(a)$ and $f(b)$ have unlike signs then an odd number of roots of $f(x) = 0$ lie between a and b (see, for example, [2]). In our case, at the lower limit:

$$g(0) = -9A^3 B^2 \tag{3.9}$$

which is strictly a negative quantity for any $B \neq 0$. ($B = 0$ is naturally excluded).
At the upper limit,

$$
\begin{aligned}
g\left(\tfrac{2-\sqrt{3}}{4}\right) \ = \ & -\tfrac{1}{256}\left(209\sqrt{3} - 362\right)\left(-16\,A^4 - 32\,A^3 - 32\,A^3\sqrt{3} + 192\,A^3 B + 96\,A^3 B\sqrt{3}\right. \\
& - \ 48\,A^2 - 48\,A^2\sqrt{3} + 480\,A^2 B + 288\,A^2 B\sqrt{3} - 1008\,A^2 B^2 \\
& - \ 576\,A^2 B^2\sqrt{3} + 72\,A + 60\,A\sqrt{3} + 48\,AB + 48\,AB\sqrt{3} - 9 + 12\,\sqrt{3})^2.
\end{aligned}
\tag{3.10}
$$

The quantity $-\tfrac{1}{256}\left(209\sqrt{3} - 362\right)$ is positive and the other quantity is a square. Therefore $g\left(\tfrac{2-\sqrt{3}}{4}\right)$ is strictly non-negative quantity irrespective of the values of A and B and is zero only when

$$
B = \frac{\tfrac{2}{3}A^2 - \tfrac{1}{3}A^2\sqrt{3} + \tfrac{1}{3}A\sqrt{3} - \tfrac{1}{3}A - \tfrac{5}{6} + \tfrac{1}{2}\sqrt{3} \pm \tfrac{1}{12}\sqrt{1 + 4A\sqrt{3} + 8A}}{A}, \tag{3.11}
$$

in which case $\left(\tfrac{2-\sqrt{3}}{4}\right)$ itself is a root of Eq.(3.7). So, for *any* boundary data (a, b), there are *an odd number of roots* within the interval $\left(0, \tfrac{2-\sqrt{3}}{4}\right]$, and hence an even number of regular bolt solutions (for the two choices of M). QED

Therefore there will at least be two Taub-Bolt-AdS infillings for any biaxial S^3 boundary. This is therefore in sharp contrast with the self-dual Taub-NUT-AdS case.

4 Region of physical interest: Isotropy and low anisotropy

The polynomial equation (3.7) is of degree seven with arbitrary coefficients determined by the values of the two radii of the Berger sphere. From Galois theory we know that general polynomial equations of degree five or above are not solvable by radicals. However, they can be solved by higher order hypergeometric functions of several variables [3]. For our purpose pursuing this line would not be illuminating. The remaining approach is to solve Eq.(3.7) numerically for p and check individually whether the solutions fall within the range $\left(0, \tfrac{2-\sqrt{3}}{4}\right]$. These steps are necessary should one seek the infilling geometries and their actions as functions of the boundary data and are left for future work. In this paper we instead count the number of possible infilling solutions without solving Eq.(3.7) numerically and see how they vary as the boundary data is varied. Indeed there are elegant ways of doing it without having to find explicit numerical solutions as will be described below.

From a physical point of view we are more interested in the qualitative behaviour when the squashing a/b is not too high, i.e., when a and b are roughly of the same order of magnitude, and they are not too small. As we will see below, it is possible to make general statements on the

possible number of infilling Taub-Bolt-AdS solution in this case. However, there are more surprises for large anisotropy which will be described in Section 5.

To proceed further we would need to recall Fourier's theorem [2]: *If $f(x)$ is a polynomial of degree n and $f_1, f_2, ...f_n$ are its successive derivatives, the number of real roots R which lie between two real numbers p and q (p < q) are such that $R \leq N - N'$, where N and N' (N $\geq$ N') respectively denote the number of changes of sign in the sequence $f_1, f_2, ...f_n$, when x = p and when x = q. Also $((N - N') - R)$ is an even number or zero.*

A formal way of proceeding would be to compute the derivatives of $g(p)$, $g'(p)$, $g''(p)$ etc. and evaluate them at zero and at $\left(\frac{2-\sqrt{3}}{4}\right)$ and check their values setting a and b roughly equal. However, for large radii and small anisotropy, it is sufficient and more illuminating to set $A = B$ in Eq.(3.7). The justification for setting them equal while considering anisotropy will be given below. With this substitution the coefficients of Eq.(3.7) simplify:

$$a_7 = 16384B^3(4B^3 - 1),$$

$$a_6 = -128(128\,B^7 + 1536\,B^6 + 112\,B^4 - 72\,B^3 - 27),$$

$$a_5 = 4(256\,B^8 + 12288\,B^7 + 55296\,B^6 + 512\,B^5 + 2016\,B^4 + 3456\,B^3 + 1728\,B + 81),$$

$$a_4 = -8\,B(384\,B^7 + 6912\,B^6 + 13824\,B^5 + 144\,B^4 - 1512\,B^3 + 1064\,B^2 - 432\,B - 81),$$

$$a_3 = 4\,B^2\left(864\,B^6 + 6912\,B^5 + 5184\,B^4 - 432\,B^3 - 1862\,B^2 - 48\,B + 81\right),$$

$$a_2 = -8\,B^3\left(216\,B^5 + 648\,B^4 - 133\,B^2 + 21\,B - 9\right),$$

$$a_1 = 3\,B^4\left(108\,B^4 + 8\,B + 21\right),$$

$$a_0 = -9\,B^5.$$

(4.1)

Note that the coefficients have been written in decreasing powers of B. It is now straightforward to check that, for B not too small, the signs of coefficients of different powers of p are insensitive to the exact value of B by observing that a_0, a_1, a_5 are positive definite for any $B > 0$ and that a_7, a_6, a_4, a_3, a_2 are positive definite for values of B (approximately) greater than 0.6299605250, 0.5123378480, 0.4031183975, 0.4565392083 and 0.4247313485 respectively. Therefore if $B > 0.6299605250$ the signs of the coefficients a_i will not change.

We now consider small anisotropy. Note that in any a_i in Eq.(4.1) above the highest two powers of B always occur with the same sign and hence the results will be unchanged if A and B are not equal and instead are roughly of the same order since the degree $(n+m)$ of the two highest powers (now of the form $A^n B^m$) will dominate. Also note that in each of the coefficients a_i in (4.1) above the coefficients of lower powers of B which have opposite signs to the highest two powers are not so large as to change the sign of a_i so long as A and B are reasonably large.

It is now easy to see that coefficients of $g(p)$ occur with alternating signs. This immediately gives the sign of the successive derivatives of $g(p)$ evaluated at zero, thus leaving us with the task

of evaluating derivatives at $\left(\frac{2-\sqrt{3}}{4}\right)$ in order to apply Fourier's theorem.

In evaluating the derivatives at $\left(\frac{2-\sqrt{3}}{4}\right)$ (henceforth denoted by q for brevity) one should check whether arguments similar to those made above can be applied to this case. The various derivatives of $g(p)$ in this case are:

$$
\begin{aligned}
g_1 = {} & 135.4256260(B + 0.4644486459)(B + 0.2555353665)(B + 0.1313663488)(B + 0.06555587950) \\
& (B - 1.141398520)(B - 1.993301852)(B^2 - 0.6177105744B + 0.1923992276),
\end{aligned}
$$

$$
\begin{aligned}
g_2 = {} & -2226.215022(B + 1.268856328)(B + 1.096797061)(B + 0.07953842687)(B - 0.4841144260) \\
& (B^2 + 0.4225153212B + 0.04767173654)(B^2 - 1.508758093\,B + 0.7095903619),
\end{aligned}
$$

$$
\begin{aligned}
g_3 = {} & 16072.86008(B + 5.568090955)(B + 0.5367072895)(B + 0.2196287557)(B + 0.1046514573) \\
& (B^2 + 0.3588828520B + 0.5018604161)(B^2 - 1.211388061B + 0.3824418783),
\end{aligned}
$$

$$
\begin{aligned}
g_4 = {} & -65496.60082(B + 13.28384331)(B + 0.3400846060)(B + 0.1452308623) \\
& (B - 0.6346052509)(B^2 + 1.920945736B + 1.048214873)(B^2 - 0.4217016508B + 0.2864195974),
\end{aligned}
$$

$$
\begin{aligned}
g_5 = {} & 122880.0(B + 37.73129542)(B + 3.820066961)(B + 0.6000369611)(B + 0.2248301376), \\
& (B^2 + 0.08583004940B + 0.2599653669)(B^2 - 0.8928401318B + 0.3309398116),
\end{aligned}
$$

$$
\begin{aligned}
g_6 = {} & -11796480.0(B + 10.13296728)(B + 0.4974325158)(B - 0.5008149571), \\
& (B^2 + 0.4893229700B + 0.2971783280)(B^2 - 0.4945521540B + 0.2811826749),
\end{aligned}
$$

$$
g_7 = 82575360\left(-1 + 4\,B^3\right)B^3.
$$

$$\tag{4.2}$$

Note that for any $B > 1.993301852$, which is true by assumption, the signs of the derivatives do not change as one goes to higher values of B. We now list the results in tabular form below:

	0	$(\frac{1}{2} - \frac{\sqrt{3}}{4})$
$g(p)$	$-$	$+$
$g_1(p)$	$+$	$+$
$g_2(p)$	$-$	$-$
$g_3(p)$	$+$	$+$
$g_4(p)$	$-$	$-$
$g_5(p)$	$+$	$+$
$g_6(p)$	$-$	$-$
$g_7(p)$	$+$	$+$
sign changes	7	6

The number of roots is the difference between the changes of sign which is one. Therefore for low squashing and not-too-small radius there will precisely be two infilling 4-geometries (corresponding to two choices of M) which contain regular bolts inside.

4.1 Round S^3

We have already mentioned that an $SO(4)$-invariant S^3 can always be filled in uniquely by parts of H^4 and with a unique self-dual Taub-NUT-AdS solution. One therefore is naturally interested in how a round S^3 can be filled in with Taub-Bolt-AdS metrics.

Lemma: *An $SO(4)$-invariant S^3 of arbitrary radius admits two Taub-Bolt-AdS infillings (for the two choices of M).*

Proof: The above analysis has already established that an $SO(4)$-invariant S^3 boundary can be filled in with two Taub-Bolt-AdS solutions assuming $B(= A) > 1.993301852$. We noted that the signs of the various derivatives of $g(p)$ at the two limits of p are not strictly the same for $B < 1.993301852$. It is not difficult to check that within the interval $(0, 1.993301852)$ the sign changes of various $g_n(0)$ are always compensated by sign changes of $g_n\left(\frac{2-\sqrt{3}}{4}\right)$ and thus the number of roots within the interval remains unchanged. It has been done numerically. We have counted the number of roots for low values of the radius with a higher degree of precision than in Eq.(4.2) which is shown in Fig. 1. Therefore a round sphere has two Taub-Bolt-AdS metrics irrespective of its radius.
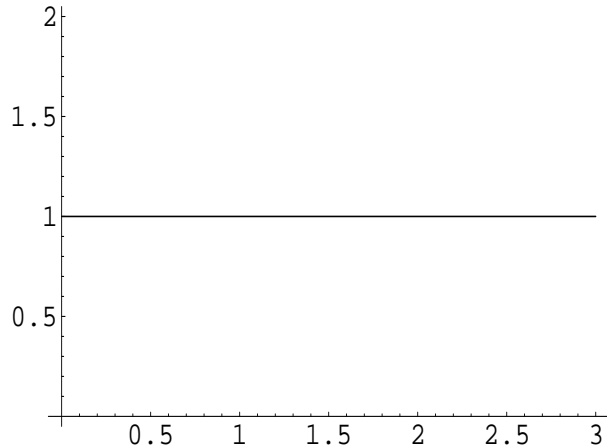

Figure 1: The number of Taub-Bolt-AdS infilling solutions (for either choice of M) is one for an $SO(4)$-invariant S^3 irrespective of it radius.

5   Anisotropy and number of infilling Taub-Bolt-AdS geometries

As mentioned in the previous section, it is only possible to make general statements for isotropic or slightly anisotropic cases. However, one is naturally interested in what happens when the Berger sphere is made more anisotropic. Does the number of solutions change or remain the same? Note that the values, and not just the ratio, of the boundary data a, b are important. This is because the presence of a cosmological constant in the interior of the Berger sphere introduces a scale.

Therefore the solutions will change if a and b are varied while their ratio is kept constant unlike the case of zero cosmological constant.

In the isotropic limit the problem can be reduced to a one dimensional one as we have seen in the previous section. We have counted the number of roots in a given interval as we varied the "parameter" B. The same procedure can be adopted for counting the number of roots as A and B are varied. Recall that the number of roots is an odd number and hence as A and B are varied the number is expected to jump in steps of two catastrophically or remain unity as in the isotropic limit. The results are plotted in Figs. 2-4 confirming the catastrophic jumps of the solutions. However, slightly to our surprise, we find that the number of roots of Eq.(3.7) can be as high as five. This is somewhat unexpected given that we are seeking solutions within a very narrow prescribed range of L.
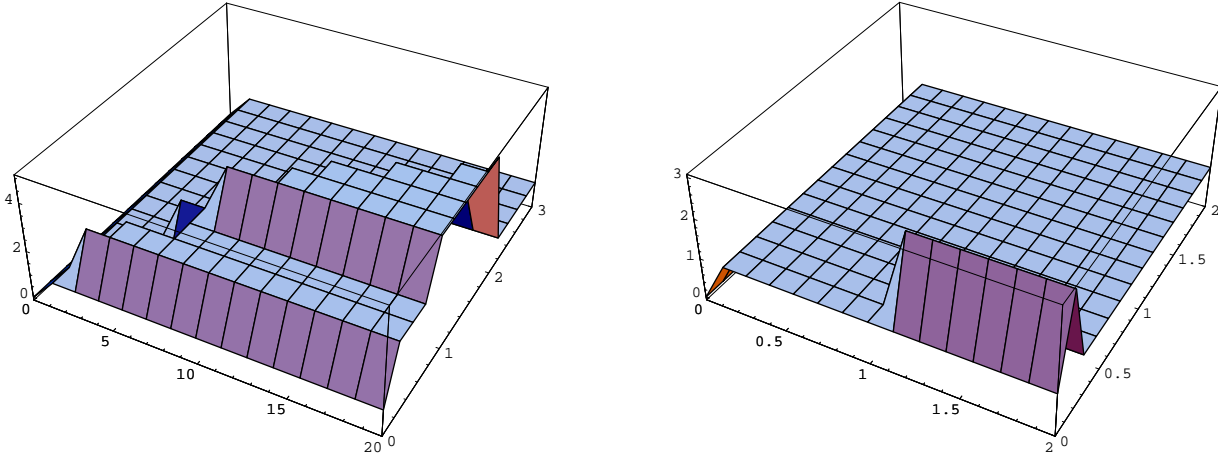

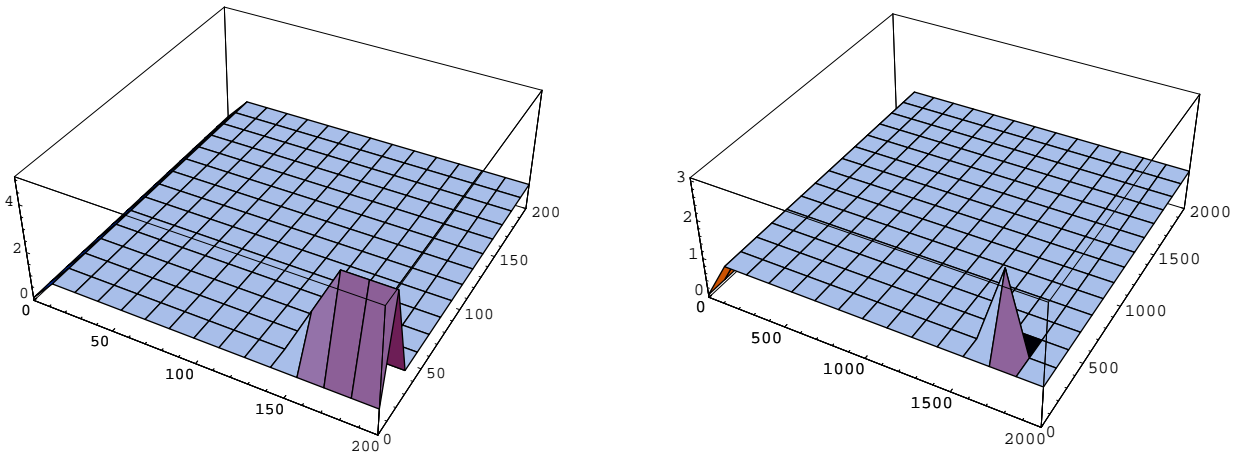Figure 2: The number of infilling solutions changes catastrophically and can be as many as five.

Figure 3: A and B of the order of 100-1000. The solutions can be as many as five even when A and B are not too small. Note that there is an averaging effect.
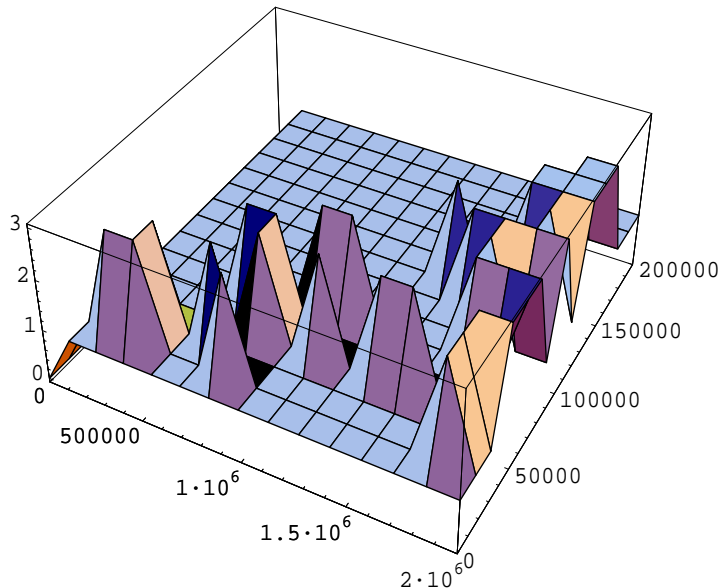
12

Figure 4: A and B very large: for $A \sim 10B$ multiple roots appear; the nature of the averaging effect in the figure means that higher number of solutions are rarer. Only in very small regions five roots will appear. Note that close to the isotropic limit the number of roots is clearly one as in Fig 3.

6 Conclusion

In this paper we have studied the filling in problem for an axially symmetric S^3 boundary with regular Taub-Bolt-AdS metrics[3]. The same boundary-value problem with self-dual Taub-NUT-AdS was studied in [1] and it was found that an S^3-boundary can be filled in with a unique or no real self-dual Taub-NUT-anti-de Sitter metric depending on whether the two radii of the S^3 satisfy the inequality (3.3) or not[4]. The Taub-Bolt-AdS metrics contain two-dimensional fixed-point sets at the centre and can be extended globally over a manifold with the topology of $\mathbb{C}P^2$. The condition of regularity of the bolt constrains the free parameter L to take values within a very narrow range on the real line. Also, as we have noted, there is a two-fold degeneracy in the family in that the other parameter M is a 1-2 function of L.

The restriction on the permissible values of L is rather stringent especially compared to the self-dual Taub-NUT-(anti-)de Sitter and the Taub-Bolt-de Sitter families for all of which L can take values freely without compromising regularity. In this case therefore one would naturally expect that such an *a priori* restriction on L would put some limits on the two radii for the Berger sphere to

---

[3]The filling in problem for a conformal boundary of S^3 topology for selfdual spaces have been studied by Pedersen [23], Hitchin [14], Tod [25]. Also the Dirichlet problem for general conformal boundaries has been studied in connection with AdS/CFT correspondence (see [20, 21] and references therein).

[4]Here we mean the self-dual Taub-NUT-(anti-)de Sitter metrics in which the parameter is non-zero and finite for which the metric is well-defined in its usual form. As explained in [1] other solutions exist as special limits to this metric which should be included if one considers the larger class of self-dual Bianchi-IX Einstein infilling solutions.

qualify for having regular Taub-Bolt-AdS infillings. However, slightly contrary to these intuitions, we found that there will always be at least one regular Taub-Bolt-AdS infilling solution (for each of the two sub-families) irrespective of the magnitude of the two radii of the Berger sphere. This is comparable to filling a round S^3 with part of H^4 which is possible irrespective of the radius of the S^3 as we have mentioned earlier. We also found that in the isotropy and low-anisotropy limits (and if the radii are not too small), the solution is unique for either of two branches. For perfect isotropy this holds for small radii as well. Furthermore, we showed that the number of such infilling solutions varies abruptly from one to five in steps of two and hence they form a catastrophe-structure in which a and b play the role of control parameters (see, for example, [24]). Compare it with Schwarzschild-AdS in four dimensions [4] in which case the boundary is the trivial bundle $S^1 \times S^2$ and can admit only two black hole solutions or no solutions depending on the two radii. We conjecture that Taub-Bolt-AdS metric (and those with other values of the integer k) and their higher dimensional generalisations are the only Einstein metrics for which such a large number of regular infilling solutions can occur. In the standard examples in the literature with Einstein metrics (including those with $\Lambda = 0$) this number, i.e., the number of infilling solutions for a given type of metrics does not exceed two. This paper provides the first example in which the number of infilling solutions exceeds two and can be as large as five, i.e., ten for the two branches put together, depending on the boundary data.

Acknowledgements

References

[1] M. M. Akbar and P. D. D'Eath, "Classical Boundary-value Problem in Riemannian Quantum Gravity and Self-dual Taub-NUT-(anti-)de Sitter Geometries," Nucl. Phys. B **648** (2003) 397 [arXiv:gr-qc/0202073].

[2] S. Barnard, and J. M. Child (1936). *Higher Algebra* (Macmillan, India).

[3] R. Birkeland, "Über die Auflösung algebraischer Gleichungen durch hypergeometrische Funktionen," Math. Zeitschrift 26 (1927) 565-578.

[4] J. D. Brown, J. Creighton and R. B. Mann, "Temperature, Energy and Heat Capacity of Asymptotically Anti-De Sitter Black Holes," Phys. Rev. D **50** (1994) 6394 [arXiv:gr-qc/9405007].

[5] B. Carter, "Hamilton-Jacobi And Schrodinger Separable Solutions Of Einstein's Equations," Commun. Math. Phys. **10** (1968) 280.

[6] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, "Large N phases, gravitational instantons and the nuts and bolts of AdS holography," Phys. Rev. D **59** (1999) 064010 [arXiv:hep-th/9808177].

[7] R. Clarkson, L. Fatibene and R. B. Mann, "Thermodynamics of (d+1)-dimensional NUT-charged AdS spacetimes," arXiv:hep-th/0210280.

[8] T. Eguchi, P. B. Gilkey and A. J. Hanson, "Gravitation, gauge theories and differential geometry," Phys. Rep. **66** (1980) 213.

[9] R. Emparan, C. V. Johnson and R. C. Myers, "Surface terms as counterterms in the AdS/CFT correspondence," Phys. Rev. D **60** (1999) 104001 [arXiv:hep-th/9903238].

[10] A. M. Ghezelbash and R. B. Mann, "Nutty bubbles," JHEP **0209** (2002) 045 [arXiv:hep-th/0207123].

[11] G. W. Gibbons and C. N. Pope, "The Positive Action Conjecture and Asymptotically Euclidean Metrics in Quantum Gravity," Commun. Math. Phys. **66** (1979) 267.

[12] S. W. Hawking, C. J. Hunter and D. N. Page, "Nut charge, anti-de Sitter space and entropy," Phys. Rev. D **59** (1999) 044033 [arXiv:hep-th/9809035].

[13] S. W. Hawking and D. N. Page, "Thermodynamics of Black Holes in Anti-De Sitter Space," Commun. Math. Phys. **87** (1983) 577.

[14] N. J. Hitchin, "Twistor Spaces, Einstein Metrics and Isomonodromic Deformations," J. Differential Geom. **42** (1995), no. 1, 30–112.

[15] G. W. Gibbons and S. W. Hawking, "Classification Of Gravitational Instanton Symmetries," Commun. Math. Phys. **66** (1979) 291.

[16] L. G. Jensen, J. Louko and P. J. Ruback, "Biaxial Bianchi Type IX Quantum Cosmology," Nucl. Phys. B **351** (1991) 662.

[17] J. M. Maldacena, "The Large N Limit of Superconformal Field Theories and Supergravity," Adv. Theor. Math. Phys. **2** (1998) 231 [Int. J. Theor. Phys. **38** (1999) 1113] [arXiv:hep-th/9711200].

[18] R. B. Mann, "Entropy of rotating Misner string spacetimes," Phys. Rev. D **61** (2000) 084013 [arXiv:hep-th/9904148].

[19] T. Sakai, "Cut loci of Berger's spheres", Hokkaido Math. J. **10** (1981), no. 1, 143-155.

[20] K. Skenderis, "Asymptotically Anti-de Sitter Spacetimes and Their Stress Energy Tensor," Int. J. Mod. Phys. A **16** (2001) 740 [arXiv:hep-th/0010138].

[21] K. Skenderis, "Lecture Notes on Holographic Renormalization," Class. Quant. Grav. **19** (2002) 5849 [arXiv:hep-th/0209067].

[22] D. N. Page, "Taub-Nut Instanton with an Horizon," Phys. Lett. B **78** (1978) 249.

[23] H. Pedersen, "Einstein metrics, spinning top motions and monopoles," Math. Ann. **274** (1986) 35.

[24] T. Poston and I. Stewart (1978). *Catastrophe Theory and its Applications* (Pitman, London).

[25] K. P. Tod, "Self-dual Einstein metrics from the Painlevé VI equation," Phys. Lett. A **190** (1994) 221.